# A *q*-Schrödinger equation based on a Hopf *q*-deformation of the Witt algebra



1999 J. Phys. A: Math. Gen. 32 4971

(http://iopscience.iop.org/0305-4470/32/26/315)

J. Phys. A: Math. Gen. **32** (1999) 4971–4981. Printed in the UK PII: S0305-4470(99)98838-8

# A $q$-Schrödinger equation based on a Hopf $q$-deformation of the Witt algebra

R Twarock

Arnold Sommerfeld Institut für Mathematische Physik, Technische Universität Clausthal, Leibnizstrasse 10, 38678 Clausthal-Zellerfeld, Germany

E-mail: reidun.twarock@tu-clausthal.de

Received 2 November 1998, in final form 25 January 1999

**Abstract.** In an earlier paper a $q$-Schrödinger equation was obtained based on a particular quantization procedure, called Borel quantization, and a related $q$-deformation of the Witt algebra. This $q$-deformation is a deformation in the category of Lie algebras and hence the corresponding $q$-Witt algebra has a trivial Hopf algebra structure. In this paper, we extend the above algebra by the addition of a set of shift-type generators, which appear in the expression for the quantum mechanical position operator and hence lead to a new type of quantum kinematics. The latter gives rise to a new kind of evolution equation and it is shown that in the limit $q \to 1$ a specific class of Schrödinger equations is obtained from it. This specification of a particular class is a new phenomenon, because in earlier references, where a different $q$-deformation has been implemented or no deformation has been used at all, such a class could not be determined uniquely. The extended algebra used here has a nontrivial Hopf structure. The appearance of the shift-type generator in the $q$-deformed picture hence leads to a selection of a particular type of dynamics and delivers in the limit $q \to 1$ new information for the characterization of the corresponding dynamics in the undeformed situation.

## 1. Introduction

A discrete model for quantum mechanics over the configuration space $S^1$ is proposed. It is a generalization of the model in [2]. The motivation for studying this type of model is twofold.

It is a common procedure in classical mechanics to derive discrete models by replacing differentials describing momentum by suitably chosen difference operators. The physical motivation for this procedure is given by the fact that the measurement of momentum in classical mechanics is related to two time-consecutive positional measurements, i.e. in mathematical terms the momentum is given as a difference operator. As a consequence, a differential describing momentum may be viewed as a mathematical idealization. Because of this, it is plausible to look for a quantization of momentum in terms of difference operators and to describe the momentum operator and the Schrödinger equation in terms of difference operators. Since the introduction of difference operators is not unique, one needs a guiding principle. Here, a quantization method, called Borel quantization, is used, which allows one to incorporate difference operators via a deformation of the Witt algebra.

The motivation for this study, which is an extension of [2, 12], is its suitability to derive in the continuous limit a Doebner–Goldin (DG) type of nonlinear Schrödinger equation with an imaginary and a real part of the nonlinearity. These DG equations are classes of nonlinear Schrödinger equations in which the imaginary part of the nonlinearity follows from the Borel quantization formalism (see e.g. [3, 5, 7, 10]), and the real part is inferred by some plausibility arguments related to the shape of the imaginary part of the nonlinearity [4]. Since the formalism presented in this contribution allows one to derive both an imaginary and a real part for the nonlinear additions to the Schrödinger equation, it can be used to justify the DG ansatz. In fact, it turns out that the imaginary part of the nonlinearity coincides with the one in the DG formalism and that the real part falls in one of the proposed classes. In this way, this study may be understood as a justification for the DG models.

As a building block for the formulation of a discrete quantum mechanics on $S^1$ $q$-derivatives, i.e. multiplicative $q$-difference operators of the form

$$\mathrm{D}_q f(z) := \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})} \tag{1}$$

are used. They can be viewed as a canonical choice for a multiplicative difference operator on $S^1$, because coordinates on $S^1$ can be expressed as $z = \mathrm{e}^{\mathrm{i}\phi}$, where $\phi \epsilon [0, 2\pi)$ parametrize the angle of the circle $S^1$, so that one has with $q^{\mathrm{i}h}$ $qz = \mathrm{e}^{\mathrm{i}(\phi+h)}$. One obtains in the limit $q \to 1$ the usual differential:

$$\mathrm{i}\mathrm{D}_q \to \mathrm{i}z\partial_z = \partial_\phi. \tag{2}$$

In order to introduce such $q$-derivatives already in the quantization method, a $q$-deformation of the algebraic structures underlying Borel quantization is used. In particular, a $q$-deformation of the Witt algebra [9, 11] is implemented. It leads to a deformation of the momentum operator in the quantum kinematics such that the latter is now given in terms of $q$-difference operators. The present $q$-deformation differs from the previous one [2] by the appearance of additional generators of shift type and we note that it has a nontrivial Hopf structure. The new generators appear in the position operator and lead to the fact that the coordinates are no longer commutative. We remark that the coproduct, as such, is not used in the formalism, but the additional set of operators, which was needed to derive such a coproduct, appear as an essential ingredient in the model and its physical interpretation.

For the derivation of a corresponding dynamics, a $q$-Schrödinger equation, an appropriately defined $q$-version of the first Ehrenfest theorem is used. It uses a symmetrization procedure, which can also be achieved by a symmetrization of the coordinates. This symmetrization is needed to guarantee that the quantum mechanical probability density is again a real quantity. The so obtained $q$-deformed dynamics is a $q$-Schrödinger equation which gives a DG equation in the limit $q \to 1$ as stated above. In contrast to the deformation used in a previous reference, the real part of the nonlinearity is specified unambiguously and only real parts of DG type appear. In particular, a particular subclass of the DG family of real parts is selected.

## 2. The quantization method

As stated in the introduction, the results of the $q$-Schrödinger equation are based on a particular quantization method, called Borel quantization. It is designed for systems localized on a continuous configuration manifold $M$, especially for $M$ with nontrivial topology as in the case of $S^1$. The kinematical part was introduced in [1, 6] and the dynamical part in [3, 5]. Since a review of Borel quantization on $S^1$ has already been given in [2], only the main results are summarized briefly.

### 2.1. The kinematics

Since classical position and momentum observables over $S^1$ can be modelled respectively via smooth real functions $f \in C^\infty(S^1, \mathbb{R})$ and smooth vector fields $X \in \mathrm{Vect}\,(S^1)$ on $S^1$, the infinite-dimensional Lie algebra

$$\mathcal{S}(S^1) := C^\infty(S^1, \mathbb{R}) \oplus \mathrm{Vect}\,(S^1) \tag{3}$$

encodes the kinematics of the system and is therefore called *kinematical algebra*.

Quantization means to construct a (quantization-) map from $\mathcal{S}(S^1)$ into the set of self-adjoint operators in the Hilbert space $\mathcal{H} = L^2(S^1, \mathrm{d}\phi)$ [8], i.e. to represent the position observables $f(\phi)$ and the momentum observables $X = X(\phi)\frac{\mathrm{d}}{\mathrm{d}\phi}$, where $f(\phi)$, $X(\phi) \in C^\infty(S^1, \mathbb{R})$, via self-adjoint operators in $\mathcal{H}$:

$$\begin{aligned} &C^\infty(S^1, \mathbb{R}) \ni f \mapsto Q(f) \in SA(\mathcal{H}) \\ &\mathrm{Vect}\,(S^1) \ni X \mapsto P(X) \in SA(\mathcal{H}). \end{aligned} \tag{4}$$

With additional physically motivated assumptions, the set of such maps can be obtained and also classified. The classification depends on the topology of $S^1$ and furthermore on a new quantum number $D \in \mathbb{R}$, which is not related to the topology.

In the Fourier picture, where $f(\phi)$ and $X(\phi) \in C^\infty(S^1, \mathbb{R})$ are given in terms of $z := \exp(\mathrm{i}\phi)$, one obtains

$$\begin{aligned} Q(f) &= \sum_{n=-\infty}^{\infty} f_n T_n \\ P(X) &= \sum_{n=-\infty}^{\infty} X_n (L_n + \mathrm{i} D n T_n) \end{aligned} \tag{5}$$

with

$$\begin{aligned} T_n &= z^n \\ L_n &= z^n \left( z \frac{\mathrm{d}}{\mathrm{d}z} + \frac{n}{2} + \theta \right) \end{aligned} \tag{6}$$

and $f_n = \bar{f}_{-n}$, $X_n = \bar{X}_{-n}$.

The generators $T_n$ and $L_n$ fulfil the commutation relations of the inhomogeneous Witt algebra $\{T_n\} \oplus \{L_n\}$:

$$\begin{aligned} &[T_m, T_n] = 0 \\ &[L_n, T_m] = m T_{m+n} \\ &[L_m, L_n] = (n - m) L_{m+n}. \end{aligned} \tag{7}$$

Unitarily inequivalent quantizations are characterized by tuples $(\theta, D)$, where $\theta \in [0, 1)$ is related to a closed one-form and hence to the topology of $S^1$. Since it is sensible to work in a particular quantization, it will be assumed that $D$ and $\theta$ are *fixed* constants throughout this paper.

### 2.2. The dynamics

The position operator $Q(f)$ and the momentum operator $P(X)$ are the building blocks of the dynamics. The connection between the kinematical situation and the dynamics is given via a generalized version of the first Ehrenfest relation. It is [5] an equation between matrix elements (Schrödinger representation) and the quantized operators $Q(f)$ and $P(X)$. For pure states $\psi \in \mathcal{H} = L^2(S^1, \mathrm{d}\phi)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \psi(\phi, t), Q(f)\psi(\phi, t)\rangle = \langle \psi(\phi, t), P(X_{\mathrm{grad}\, f})\psi(\phi, t)\rangle \qquad \forall f \in C^\infty(S^1, \mathbb{R}) \tag{8}$$

where $X_{\mathrm{grad}\, f} = f'(\phi)\frac{\mathrm{d}}{\mathrm{d}\phi}$.

It leads to a Fokker–Planck-type equation for the time derivative of the probability density for positions $\rho(\phi, t) = \bar{\psi}(\phi, t)\psi(\phi, t)$ [3]

$$\dot{\rho} = \frac{\mathrm{i}}{2}(\bar{\psi}\psi'' - \bar{\psi}''\psi) + D\rho'' - \theta\rho' \tag{9}$$

and thus restricts the evolution equation for pure states $\psi$.

This can be seen with the ansatz

$$\begin{aligned} \mathrm{i}\partial_t\psi &= H\psi + G[\bar{\psi}, \psi]\psi \\ -\mathrm{i}\partial_t\bar{\psi} &= \bar{H}\bar{\psi} + \bar{G}[\bar{\psi}, \psi]\bar{\psi} \end{aligned} \tag{10}$$

where $H$ is a linear operator, later interpreted as Hamiltonian, and $G[\bar{\psi}, \psi] \equiv \mathrm{Re}\, G[\bar{\psi}, \psi] + \mathrm{i}\mathrm{Im}\, G[\bar{\psi}, \psi]$ a nonlinear function of $\bar{\psi}, \psi$ (possibly also on $t$ and $\phi$), because it allows one to express $\dot{\rho}$ as

$$\dot{\rho} = \dot{\bar{\psi}}\psi + \bar{\psi}\dot{\psi} = \mathrm{i}(\psi(\bar{H}\bar{\psi}) - \bar{\psi}(H\psi)) + 2\mathrm{Im}\, G[\bar{\psi}, \psi]\rho. \tag{11}$$

Together with (9) it leads to an expression for the Schrödinger equation ($m = 1, \hbar = 1$):

$$\mathrm{i}\partial_t\psi = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\psi - \mathrm{i}\theta\frac{\mathrm{d}}{\mathrm{d}\phi}\psi + \mathrm{i}\frac{D}{2\rho}\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\rho\right)\psi + \underbrace{\mathrm{Re}\, G[\bar{\psi}, \psi]}_{R[\psi]}\psi \tag{12}$$

in which the real part $R[\psi]$ of the nonlinear term $G[\bar{\psi}, \psi]$ remains undetermined as a consequence of the fact that the real part of the nonlinearity does not appear in (11).

In order to also specify the real part of the nonlinearity, further assumptions are necessary. Motivated by the shape of the imaginary part of $G[\bar{\psi}, \psi]$, one demands a number of plausible requirements [3]:

(i) $R[\psi]$ should be proportional to $D$, i.e. vanishing for $D = 0$.

(ii) $R[\psi]$ should have derivatives no higher than of second order and occurring only in the numerator.

(iii) $R[\psi]$ should be complex homogeneous of degree zero, i.e. $R[\alpha\psi] = R[\psi]$ for all $\alpha \epsilon \mathbb{C}$.

These yield a family of possible real parts, the DG-classes of real parts:

$$R[\psi] := D_1\frac{j_0'}{\rho} + D_2\frac{\rho''}{\rho} + D_3\frac{j_0^2}{\rho^2} + D_4\frac{(j_0\rho')}{\rho^2} + D_5\frac{(\rho')^2}{\rho^2}. \tag{13}$$

It is stressed already here that the $q$-Schrödinger equation derived in this contribution will allow for a specification of a fixed real part of the nonlinearity in the limit $q \to 1$, which lies in this DG-class of real parts.

## 3. $q$-deformation of the kinematics

In this section we review the $q$-deformation of the Witt algebra used earlier and show how it can be augmented, by the addition of further generators, to a nontrivial Hopf algebra [9, 11]. The corresponding generators are then implemented to obtain a $q$-deformed version of the quantum Borel kinematics on $S^1$.

### 3.1. The $q$-Witt algebra as a Hopf algebra

Previously, a $q$-deformation of the Witt algebra of the following form has been used:

$$[\mathcal{L}_m^{(j_1,\theta)}, \mathcal{L}_n^{(j_2,\theta)}] = \frac{[j_1\frac{n}{2} - j_2\frac{m}{2}][j_1 + j_2]}{[j_1][j_2]}\mathcal{L}_{m+n}^{(j_1+j_2,\theta)} + \frac{[j_1\frac{n}{2} + j_2\frac{m}{2}][j_2 - j_1]}{[j_1][j_2]}\mathcal{L}_{m+n}^{(j_2-j_1,\theta)}. \tag{14}$$

For $q \to 1$, the commutation relations reproduce the commutation relations of the Witt algebra. The extra parameter $j$, which has been introduced during the deformation, vanishes in the limit $q \to 1$. The $q$-Witt algebra (14) is realized by

$$\mathcal{L}_m^{(j,\theta)} := z^m \frac{[j(N_z + \frac{m}{2} + \theta)]}{[j]}. \tag{15}$$

Here, $N_z$ is an abbreviation for $z\partial_z$.

The $q$-Witt algebra is again a Lie algebra. However, it is possible, via the introduction of a set of additional operators, to find a larger algebra with nontrivial Hopf structure containing the $q$-Witt algebra above [9, 11].

In particular, in [11] a set of shift operators $K_s$ of the form

$$K_s := q^{sN_z} \tag{16}$$

is introduced, which is commutative, i.e.

$$[K_s, K_r] = 0 \qquad \forall r, s \tag{17}$$

and couples to the generators of the $q$-Witt algebra (14) as follows:

$$K_s\mathcal{L}_m^{(j,\theta)} = q^{sm}\mathcal{L}_m^{(j,\theta)}K_s \qquad \forall s, m, j. \tag{18}$$

The augmented algebra (14), (17) and (18) is now a nontrivial Hopf algebra with the following coproduct $\Delta$, counit $\epsilon$ and antipode $\gamma$ [11]:

$$\begin{aligned}
&\Delta(\mathcal{L}_m^{(j,\theta)}) = \mathcal{L}_m^{(j,\theta)} \otimes K_m + K_m \otimes \mathcal{L}_m^{(j,\theta)} \qquad \Delta(K_l) = K_l \otimes K_l \\
&\epsilon(\mathcal{L}_m^{(j,\theta)}) = 0 \qquad \epsilon(K_l) = 1 \\
&\gamma(\mathcal{L}_m^{(j,\theta)}) = -K_m^{-1}\mathcal{L}_m^{(j,\theta)}K_m^{-1} \qquad \gamma(K_l) = K_l^{-1}.
\end{aligned} \tag{19}$$

### 3.2. $q$-deformation of position and momentum operators on $S^1$

Based on the $q$-Witt algebra (14) a $q$-deformation of the quantum Borel kinematics on $S^1$ had been introduced as†:

$$\begin{aligned}
Q_q(f) &= \sum_{n=-\infty}^{\infty} f_n T_n (= Q(f)) \\
P_q^j(X) &= \sum_{n=-\infty}^{\infty} X_n \left(\mathcal{L}_n^{(j,\theta)} + \mathrm{i}\frac{[jn]}{[j]} D T_n\right).
\end{aligned} \tag{20}$$

In these expressions, only the momentum operator is deformed, whereas the position operator remains unaffected by the deformation procedure. In particular, via deformation, the momentum operator becomes dependent on an additional parameter $j$, whereas the position operator, remaining undeformed, is not equipped with such an additional freedom.

This causes an imbalance in the treatment of position and momentum operator. In this contribution this problem is cured with the help of the additional generators $K_s$, which were introduced in the previous section. These generators are also used to deform the position

† Note that instead of $[n]$ in [2] the expression $\frac{[jn]}{[j]}$ is used as in [12].

operators in (20) and the index $s$ will then be the counterpart to the index $j$ for the momentum operator.

Since $K_s$ is a shift operator, one has $K_s \to 1$ for $q \to 1$, so that a multiplication with $K_s$ does not affect the corresponding identities in this limit. The following expression, denoted as Hopf algebra quantum Borel kinematics, is hence a possible way to introduce the operators $K_s$ into the $q$-quantum Borel kinematics (20):

$$\begin{aligned} Q_q^s(f) &= \sum_{n=-\infty}^{\infty} f_n T_n K_s \\ P_q^j(X) &= \sum_{n=-\infty}^{\infty} X_n \left( \mathcal{L}_n^{(j,\theta)} + \mathrm{i} \frac{[jn]}{[j]} D T_n \right). \end{aligned} \tag{21}$$

It differs from (20) by the factor $K_s$ in the position operator. As a consequence, the discrete basis of the coordinate part $B_{n,s} := T_n K_s$ ceases to be Abelian, and one obtains the following for the coordinate part instead of the first equation in formula (7):

$$[B_{m,s}, B_{n,j}] = (q^{sn} - q^{jm}) B_{n+m,s+j}. \tag{22}$$

This means that we have a non-commutative discrete geometry which appears as a consequence of Borel quantization and the new set of generators $K_s$ in the deformation of the Witt algebra. The coupling of the new coordinates $B_{n,s}$ to the generators $\mathcal{L}_n^{(j,\theta)}$ is given by the relation

$$q^{-sn} B_{m,s} \mathcal{L}_n^{(j,(\theta+m))} = \mathcal{L}_n^{(j,\theta)} B_{m,s} \tag{23}$$

which generalizes the relation in [2] for the commutative coordinates $T_n$ ($T_m \mathcal{L}_n^{(j,(\theta+m))} = \mathcal{L}_n^{(j,\theta)} T_m$). The part which only involves the generators $\mathcal{L}_n^{(j,\theta)}$ is given by (14) as before.

It is possible to introduce a symmetrization on the coordinates of the form

$$S_{m,s} := \tfrac{1}{2}(T_m K_s + K_s T_m) \tag{24}$$

which like the product $T_m K_s$ lead to the generators $T_m$ in the limit $q \to 1$. In the next section it turns out that such a symmetrized form of the coordinates can be used instead of a symmetrization of the Ehrenfest relation as indicated in (28).

Finally, we remark that for the particular choice of the deformation parameter $q$ as a root of unity, one can restrict the configuration manifold $S^1$ to its $N$-point discretization

$$S_N^1 := \left\{ z_l = \exp\left(\frac{2\pi \mathrm{i}}{N} l\right) \middle| l = 0, \ldots, N-1 \right\}. \tag{25}$$

In this case the action of $\mathrm{i}[N_z]$ on the wavefunctions $\psi$ (now spanning a finite-dimensional Hilbert space $\mathcal{H}_N$ of sequences) of the form

$$\psi(l) = \sum_{n=0}^{N-1} \psi_{nl} z_l^n \tag{26}$$

is given by $\mathrm{i}[N_z]\psi(l) = \frac{\mathrm{i}}{q-q^{-1}} \sum_n \psi_{n,l}((qz_l)^n - (q^{-1}z_l)^n)$ and is hence well defined on $S_N^1$, because $qz_l$ and $q^{-1}z_l$ are again lattice points. Correspondingly, only a finite number of generators for the generalized coordinates arise, so that only finite sums appear in (21).

In this discrete setting, the parameter $j$ in $\mathcal{L}_n^{(j,\theta)}$ and $s$ in $K_s$ have the following interpretation: $\mathcal{L}_n^{(j,\theta)}$ contains—dependent on $j$—differences between different points of $S_N^1$, e.g. between next nearest neighbours for $j = 2$ or even further points, i.e. it measures how coarse grained the discretization is. Similarly, $s$ is a measure for how coarse grained the jumps initiated by the shift-operators $K_s$ are.

## 4. $q$-deformation of the dynamics

It is apparent from the construction in the undeformed case that an appropriately generalized version of the first Ehrenfest relation is the main building block for the derivation of a dynamics. In order to adapt (8) to the situation of the Hopf algebra quantum Borel kinematics, we need a symmetrization procedure on the expectation values in the first Ehrenfest relation. In particular, if $\langle\,,\,\rangle$ denotes the usual scalar product as in [2] and $S$ denotes a shift operator, we need to use expressions of the form

$$\langle\psi, S\phi\rangle_{sym} := \tfrac{1}{2}\{\langle\psi, S\phi\rangle + \langle\bar{S}\psi, \phi\rangle\}. \tag{27}$$

It means taking the usual scalar product in a first step and then symmetrizing the dependence on the shift operator $S$. This procedure is necessary because of the occurrence of the operators $K_s$ in the position operators in (21) in order to guarantee that the deformed counterpart to the quantum mechanical probability density $\rho = \bar{\psi}\psi$ is a real quantity. Equivalently, the symmetrized coordinates (24) can be used instead of the coordinates in (21) to obtain the same effect.

In the symmetrized version, the first Ehrenfest relation reads:

$$\partial_t\langle\psi(t,z), Q_q^s(f)\psi(t,z)\rangle_{sym} = \langle\psi(t,z), P_q^j(X_{\mathrm{grad}_q f})\psi(t,z)\rangle_{sym} \tag{28}$$

where

$$P_q^j(X_{\mathrm{grad}_q f}) = \sum_n \mathrm{i}\frac{[nj]}{[j]} f_n \left(\mathcal{L}_n^{(j,\theta)} + \mathrm{i}\frac{[nj]}{[j]} DT_n\right) \tag{29}$$

as in [12]. It is such that it reproduces (8) in the limit $q \to 1$.

We start by calculating the left- and right-hand side of (28) separately. The left-hand side yields:

$$\begin{aligned}\partial_t\langle\psi(t,z), Q_q^s(f)\psi(t,z)\rangle_{sym} &= \partial_t\langle\psi(t,z), f(z)K_s\psi(t,z)\rangle_{sym}\\ &= \partial_t\tfrac{1}{2}\{\langle\psi, f(K_s\phi)\rangle + \langle(K_s\psi), f\phi\rangle\}\\ &= \langle f, \partial_t\tfrac{1}{2}(\bar{\psi}(K_s\psi) + (K_s\bar{\psi})\psi)\rangle\\ &=: \langle f, A\rangle\end{aligned} \tag{30}$$

where the real-valued functions $f$ do not depend on $t$ (compare with the undeformed case in section 2.2) and are hence unaffected by the operator $\partial_t$. The last line in (30) contains the usual scalar product. The $q$-analogue of the quantum mechanical probability density $\rho = \bar{\psi}\psi$ is hence given as $\frac{1}{2}(\bar{\psi}(K_s\psi) + (K_s\bar{\psi})\psi)$. Since the shift-operators $K_s$ are real, it is a real quantity.

Correspondingly, the right-hand side of (28) leads to an expression of the form $\langle f, B\rangle$ in the usual scalar product with†

$$\begin{aligned}B := \frac{\mathrm{i}}{2[j]^2}\Bigg\{&\left([jN_z]\left[j\frac{N_z}{2}\right]\bar{\psi}\right)(q^{-\varepsilon_1 jN_z+\varepsilon_2 j\frac{N_z}{2}}\psi) - \left([jN_z]\left[j\frac{N_z}{2}\right]\psi\right)(q^{\varepsilon_1 jN_z+\varepsilon_2 j\frac{N_z}{2}}\bar{\psi})\\ &-([jN_z]q^{\varepsilon_2 j\frac{N_z}{2}}\bar{\psi})\left(\left[j\frac{N_z}{2}\right]q^{-\varepsilon_1 jN_z}\psi\right) + ([jN_z]q^{\varepsilon_2 j\frac{N_z}{2}}\psi)\left(\left[j\frac{N_z}{2}\right]q^{\varepsilon_1 jN_z}\bar{\psi}\right)\\ &+\left([jN_z]\left[j\frac{N_z}{2}\right]q^{\varepsilon_2 j\frac{N_z}{2}}\bar{\psi}\right)(q^{-\varepsilon_1 jN_z}\psi) - \left([jN_z]\left[j\frac{N_z}{2}\right]q^{\varepsilon_2 j\frac{N_z}{2}}\psi\right)(q^{\varepsilon_1 jN_z}\bar{\psi})\end{aligned}$$

† The change of $[N_z]$ into $\frac{[jN_z]}{[j]}$ and $\varepsilon$ into $j\varepsilon$ with respect to the expression in [2] are rooted in the corresponding changes in (20). It is the convention used in [12].

$$- ([jN_z]\bar{\psi}) \left( \left[ j\frac{N_z}{2} \right] q^{-\varepsilon_1 jN_z + \varepsilon_2 j\frac{N_z}{2}} \psi \right) + ([jN_z]\psi) \left( \left[ j\frac{N_z}{2} \right] q^{\varepsilon_1 jN_z + \varepsilon_2 j\frac{N_z}{2}} \bar{\psi} \right) \Bigg\}$$
$$- \frac{\mathrm{i}}{[j]^2} D([jN_z]^2 \bar{\psi}\psi) \tag{31}$$

where $q^{aN_z}$ acts as a shift-operator on functions $f(z)$, i.e.

$$q^{aN_z} f(z) = f(q^a z). \tag{32}$$

We restrict ourselves to the case of $\theta = 0$ to keep the argument clear. Furthermore, the $\theta$-dependence is not essential for the derivation of the nonlinear parts, so that it is not crucial for the main result, which is the derivation of possible real parts.

In terms of $\psi_1$ and $\psi_2$, where $\psi = \psi_1 + \mathrm{i}\psi_2$, one finds for the $D$-independent part:

$$\begin{aligned} B = \frac{1}{2[j]^2} \Bigg\{ & \left( -[jN_z] \left[ j\frac{N_z}{2} \right] \psi_1 \right) ((q^{\varepsilon_1 jN_z} + q^{-\varepsilon_1 jN_z}) q^{\varepsilon_2 j\frac{N_z}{2}} \psi_2) \\ & + \left( [jN_z] \left[ j\frac{N_z}{2} \right] \psi_2 \right) ((q^{\varepsilon_1 jN_z} + q^{-\varepsilon_1 jN_z}) q^{\varepsilon_2 j\frac{N_z}{2}} \psi_1) \\ & + (-[jN_z] q^{\varepsilon_2 \frac{j}{2} N_z} \psi_2) \left( \left[ j\frac{N_z}{2} \right] (q^{\varepsilon_1 jN_z} + q^{-\varepsilon_1 jN_z}) \psi_1 \right) \\ & + ([jN_z] q^{\varepsilon_2 \frac{j}{2} N_z} \psi_1) \left( \left[ j\frac{N_z}{2} \right] (q^{\varepsilon_1 jN_z} + q^{-\varepsilon_1 jN_z}) \psi_2 \right) \\ & \left( -[jN_z] \left[ j\frac{N_z}{2} \right] q^{\varepsilon_2 j\frac{N_z}{2}} \psi_1 \right) ((q^{\varepsilon_1 jN_z} + q^{-\varepsilon_1 jN_z}) \psi_2) \\ & + \left( [jN_z] \left[ j\frac{N_z}{2} \right] q^{\varepsilon_2 j\frac{N_z}{2}} \psi_2 \right) ((q^{\varepsilon_1 jN_z} + q^{-\varepsilon_1 jN_z}) \psi_1) \\ & + (-[jN_z]\psi_2) \left( \left[ j\frac{N_z}{2} \right] (q^{\varepsilon_1 jN_z} + q^{-\varepsilon_1 jN_z}) q^{\varepsilon_2 \frac{j}{2} N_z} \psi_1 \right) \\ & + ([jN_z]\psi_1) \left( \left[ j\frac{N_z}{2} \right] (q^{\varepsilon_1 jN_z} + q^{-\varepsilon_1 jN_z}) q^{\varepsilon_2 \frac{j}{2} N_z} \psi_2 \right) \Bigg\}. \end{aligned} \tag{33}$$

Fulfilling the generalized Ehrenfest relation (28) means equating the expressions for $A$ and $B$. This restricts possible evolution equations for $\psi$ and like this contains information on $\mathrm{i}\partial_t\psi$. To extract this information, a general ansatz for such an evolution equation has to be made. We use an ansatz with $H_q^j$ linear in $\psi$ and $G_q^j[\bar{\psi}, \psi]$ nonlinear in $\psi$, $\bar{\psi}$:

$$\begin{aligned} \mathrm{i}\partial_t \psi \bar{\psi} &= (H_q^j \psi)(S\bar{\psi}) + (G_q^j[\bar{\psi}, \psi]\psi)(R\bar{\psi}) \\ -\mathrm{i}\partial_t \bar{\psi} \psi &= (\bar{H}_q^j \bar{\psi})(\bar{S}\psi) + (\bar{G}_q^j[\bar{\psi}, \psi]\bar{\psi})(\bar{R}\psi). \end{aligned} \tag{34}$$

$S$ and $R$ are shift operators like in (32), which typically occur in a $q$-deformed theory. Equations (34) reduces to (10) in the limit $q \to 1$, if $H_q^j$ and $G_q^j[\bar{\psi}, \psi]$ are such that they give $H$ and $G[\bar{\psi}, \psi]$ in this limit.

With (34) we get

$$\begin{aligned} A &= \tfrac{1}{2}((\partial_t K_s \bar{\psi})(\psi) + (\partial_t \psi)(K_s \bar{\psi}) + (\partial_t \bar{\psi})(K_s \psi) + (\partial_t K_s \psi)(\bar{\psi})) \\ &= \mathrm{i}\{(\bar{H}_q^j K_s \bar{\psi})(\bar{S}\psi) + (\bar{G}_q^j K_s \bar{\psi})(R\psi) - (H_q^j K_s \psi)(S\bar{\psi}) - (G_q^j K_s \psi)(R\bar{\psi}) \\ &\quad + (\bar{H}_q^j \bar{\psi})(\bar{S} K_s \psi) + (\bar{G}_q^j \bar{\psi})(R K_s \psi) - (H_q^j \psi)(S K_s \bar{\psi}) - (G_q^j \psi)(R K_s \bar{\psi})\}. \end{aligned} \tag{35}$$

Now assuming that the linear operator $H_q^j$, is denoted by $H$, and the corresponding shift $S$ are real and expressing the identity again via $\psi = \psi_1 + \mathrm{i}\psi_2$ in terms of $\psi_1$ and $\psi_2$, we obtain that the terms on the right-hand side of (35) involving the linear terms $H$ are given by:

$$(H\psi_2)(SK_s\psi_1) + (HK_s\psi_2)(S\psi_1) - (H\psi_1)(SK_s\psi_2) - (HK_s\psi_1)(S\psi_2). \tag{36}$$

Equating this expression with the linear terms in (33), we find

$$H = \frac{1}{[j]^2}[jN_z]\left[j\frac{N_z}{2}\right] \qquad S = \tfrac{1}{2}(q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z}) \qquad K_s = q^{\varepsilon_2 j\frac{N_z}{2}}. \tag{37}$$

In particular, since $K_s = q^{sN_z}$, the parameter $s$ is fixed in dependence on $j$. The formalism to derive a $q$-Schrödinger equation corresponding to the ansatz (34) hence requires the selection $s = \frac{\varepsilon_2 j}{2}$. Reinserting this into the Hopf algebra quantum Borel kinematics means that the freedom in the position and momentum operator is not independent and linked precisely by this condition on $s$ via the dynamics.

The resulting linear part of the $q$-Schrödinger equation hence is ($j \in 2\mathbb{N}$, $\varepsilon_1, \varepsilon_2 = \pm 1$):

$$(H_q^j\psi)(S\bar{\psi}) = \left([jN_z]\left[j\frac{N_z}{2}\right][j]^{-2}\psi\right)\frac{1}{2}(q^{\varepsilon_1 N_z} + q^{-\varepsilon_1 N_z})\bar{\psi}. \tag{38}$$

It leads for all $j$ in the limit $q \to 1$ to the Hamiltonian $-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}$ obtained in section 2.2. Furthermore, it resembles the Hamiltonian obtained with the more restricted deformation up to a factor $q^{\varepsilon_2 j\frac{N_z}{2}}$, which now is compensated by the operator $K_s$.

Up to now, the nonlinear terms, $G_q^j$, have been neglected. In order to derive them, the terms in (35), which contain a dependence on $G_q^j[\bar{\psi}, \psi] := G_1 + \mathrm{i}G_2$, are collected. The imaginary terms cancel as expected and one obtains with the notation $R = R_1 + \mathrm{i}R_2$, $\psi$ the following real quantity:

$$G_1 h_1(\psi_1, \psi_2, R_1, R_2, K_s) + G_2 h_2(\psi_1, \psi_2, R_1, R_2, K_s) \tag{39}$$

where

$$\begin{aligned} h_1(\psi_1, \psi_2, R_1, R_2, K_s) = {} & (K_s\psi_1)(R_2\psi_1) + (K_s\psi_2)(R_2\psi_2) \\ & + (K_s\psi_2)(R_1\psi_1) - (K_s\psi_1)(R_1\psi_2) + (\psi_1)(R_2K_s\psi_1) + (\psi_2)(R_2K_s\psi_2) \\ & + (\psi_2)(R_1K_s\psi_1) - (\psi_1)(R_1K_s\psi_2) \end{aligned} \tag{40}$$

and

$$\begin{aligned} h_2(\psi_1, \psi_2, R_1, R_2, K_s) = {} & (K_s\psi_1)(R_1\psi_1) + (K_s\psi_2)(R_1\psi_2) \\ & + (K_s\psi_1)(R_2\psi_2) - (K_s\psi_2)(R_2\psi_1) + (\psi_1)(R_1K_s\psi_1) + (\psi_2)(R_1K_s\psi_2) \\ & + (\psi_1)(R_2K_s\psi_2) - (\psi_2)(R_2K_s\psi_1). \end{aligned} \tag{41}$$

It now has to be equated with the nonlinear terms in (33). The corresponding identity implicitly contains $G_1$, $G_2$, $R_1$ and $R_2$. Depending on the shifts $R_1$ and $R_2$, $G_1$ and $G_2$ can be obtained from it and yield the nonlinear terms via

$$F_{NL} \equiv (G_q^j[\bar{\psi}, \psi]\psi)(R\bar{\psi}) = ((G_1 + \mathrm{i}G_2)\psi)(R_1 + \mathrm{i}R_2)\bar{\psi}. \tag{42}$$

In particular, the nonlinear $q$-Schrödinger equation is then given as ($j \in 2\mathbb{N}$, $\varepsilon_1, \varepsilon_2 = \pm 1$):

$$\mathrm{i}(\partial_t\psi)\bar{\psi} = (H_q^j\psi)(S\bar{\psi}) + F_{NL} \tag{43}$$

with $H_q^j$ and $S$ as in (38).

It is important to stress that the nonlinear term $F_{NL}$ depends on the choices of $R_1$ and $R_2$, and indeed, for different choices of $R_1$ and $R_2$ different types of nonlinear terms are obtained.

The limit $q \to 1$ of the linear part of the evolution equation (38) has already been discussed above. In order to obtain the nonlinear part in the limit $q \to 1$, it is necessary to expand the functions $h_1(\psi_1, \psi_2, R_1, R_2, K_s)$ and $h_2(\psi_1, \psi_2, R_1, R_2, K_s)$ as well as the nonlinear terms in (33) in leading orders of $h$, where $h$ is given as $q = \mathrm{e}^h$. For the nonlinear terms in (33) one finds $A^{(1)} + A^{(2)}$ with

$$\begin{aligned} A^{(1)} &= \frac{j^2}{8\mathrm{i}}(\bar{\psi}'\psi''' - \psi'\bar{\psi}''')h^2 + \mathrm{O}(h^3) \\ A^{(2)} &= -D\rho'' + \mathrm{O}(h). \end{aligned} \tag{44}$$

For $h_1(\psi_1, \psi_2, R_1, R_2, K_s)$ one derives with the notation $R = R_1 + \mathrm{i}R_2 = (aq^{\alpha N_z} + bq^{\beta N_z}) + \mathrm{i}(cq^{\gamma N_z} + dq^{\delta N_z})$ ($a, b, c, d, \alpha, \beta, \gamma$ and $\delta$ are real constants) the expression

$$\begin{aligned} h_1(\psi_1, \psi_2, R_1, R_2, K_s) &= 2(c+d)(\psi_1^2 + \psi_2^2) + 2h\left((c+d)\varepsilon_2\frac{j}{2} + c\gamma + d\delta\right) \\ &\quad \times(\psi_1\psi_1' + \psi_2\psi_2') + (a\alpha + b\beta)(\psi_2\psi_1' - \psi_1\psi_2') + \mathrm{O}(h^2) \end{aligned} \tag{45}$$

and a similar expression for $h_2(\psi_1, \psi_2, R_1, R_2, K_s)$ which follows from (45) using $h_2(\psi_1, \psi_2, R_1, R_2, K_s) = h_1(\psi_1, \psi_2, R_2, R_1, K_s)$.

An implementation of these expansions shows that in the limit $q \to 1$, one always obtains the same imaginary part for the nonlinear functional, which has also been derived without $q$-deformation. In addition, a particular class of real parts occurs:

$$\frac{D\rho''}{2\rho}. \tag{46}$$

It is equal to the imaginary part obtained in the limit $q \to 1$ or without deformation in the framework of Borel quantization. For $R = 1$ (trivial shift operator) the real part remains undetermined like before, as expected for consistency.

It is interesting to remark that the deformation in the earlier reference has led to different classes of real parts. The reason for this discrepancy lies in the fact that here one has obtained nonlinear terms in leading orders of $h^2$ in (44), whereas the counterpart to this formula in [12] (formula (7.48)) is given in leading orders of $h$, which have cancelled each other here due to the contributions from the symmetrization.

In this way, the formalism presented here contains more information and leads to more specific results. It is interesting to note that the real part derived here lies in the DG-class of real parts introduced in section 2. In contrast to this, there are two nontrivial classes of real parts for the more restricted deformation, one of which coincides with the class derived here, and another one, which does not fall into the DG-classes. The deformation procedure used here hence has picked from these possibilities precisely this class, which is a member of the DG-classes.

Finally, the set of $q$-Schrödinger equations (43) is indicated directly as a difference equation in dependence of $j$:

$$\begin{aligned} &(\mathrm{i}\partial_t\psi(l))\bar{\psi}(l) = \frac{1}{2(q^j - q^{-j})^2} \\ &\left(\psi\left(l + \frac{3j}{2}\right) - \psi\left(l - \frac{j}{2}\right) - \psi\left(l + \frac{j}{2}\right) + \psi\left(l - \frac{3j}{2}\right)\right) \\ &(\bar{\psi}(l+j) + \bar{\psi}(l-j)) + F_{NL}. \end{aligned} \tag{47}$$

It displays a more symmetric structure than the previous result.

## 5. Summary

A method introduced in an earlier paper for the derivation of a discrete quantum mechanics on $S^1$ has been extended in such a way, that the quantum kinematics is now given in terms of generators of an augmented algebra. This change in the kinematical structure has led, together with an appropriate generalization of the first Ehrenfest theorem, to a new $q$-Schrödinger equation. In particular, in the limes $q \to 1$, a specific type of real part for the nonlinearity in the Schrödinger equation could be derived for the first time, whereas no information is obtained in the framework of Borel quantization without $q$-deformation, or only restricted information with the previous deformation. In particular, a unique class of real parts has been specified, and it is interesting that a member of the DG-class of real parts has been selected. The formalism presented here has hence led to a particular type of nonlinear dynamics on the quantum level.

## Acknowledgments

This work was supported by the ministry of Lower Saxony in the framework of the Dorothea-Erxleben programme. I am very grateful to Professor H D Doebner and Professor V K Dobrev for their continuing support and many helpful discussions during the course of this work. Furthermore, I would like to thank the referees for valuable comments.

## References

[1] Angermann B, Doebner H D and Tolar J 1983 Quantum kinematics on smooth manifolds (*Lecture Notes im Maths vol 1037*) (Berlin: Springer) pp 171–208
[2] Dobrev V K, Doebner H D and Twarock R 1997 A discrete, nonlinear $q$-Schrödinger equation via Borel quantization and $q$-deformation of the Witt algebra *J. Phys. A: Math. Gen.* **38** 1161–82
[3] Doebner H D and Goldin G A 1992 On a general nonlinear Schrödinger equation admitting diffusion currents *Phys. Lett.* A **162** 397–401
[4] Doebner H D and Goldin G A 1994 Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations *J. Phys. A: Math. Gen.* **27** 1771–80
[5] Doebner H D and Hennig J D 1995 A quantum mechanical evolution equation for mixed states from symmetry and kinematics *Symmetries in Science VIII* ed B Gruber (New York: Plenum) pp 85–90
[6] Doebner H D and Müller U A 1993 Borel kinematics of rank $k$ on smooth manifolds *J. Phys. A: Math. Gen.* **26** 719–30
[7] Doebner H D and Nattermann P 1996 Borel quantization: kinematics and dynamics *Acta Phys. Pol.* B **27** 2327–39
Doebner H D and Nattermann P 1996 Borel quantization: kinematics and dynamics *Acta Phys. Pol.* B **27** 4003
[8] Doebner H D and Tolar J 1990 Infinite-dimensional symmetries *Ann. Phys., Lpz.* **47** 116–22
[9] Hiro-Oka H, Matsui O, Naito T and Saito S 1990 On the $q$-deformation of Virasoro algebra *TMUP-HEL-9004*
[10] Nattermann P, Scherer W and Ushveridze A G 1994 Exact solutions of the general Doebner–Goldin equation *Phys. Lett.* A **184** 234–40
[11] Saito S 1991 $q$-Virasoro algebra and $q$-strings *Quarks, Symmetries and Strings* ed M Kaku *et al* (New York: World Scientific) pp 231–40
[12] Twarock R 1997 Quantum mechanics on $S^1$ with $q$-difference operators *PhD Thesis* Technische Universität, Clausthal